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ON MINIMAL OBSERVATIONS IN A GAME OF ENCOUNTER

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We consider the differential game of the encounter of "isotropic rockets" [1]. Its solution, under the condition of complete informativeness of the players, has been constructed in [2]. We investigate the question of the minimal information needed by the players to realize a saddle situation. The statement of similar game problems with incomplete information has been given in [3].

1. Let the motion of players X and Y on a fixed time interval $[0, T]$, $T > 0$ be specified by the relations

$$X: x_1^{\cdot} = x_2, \quad x_2^{\cdot} = u, \quad |u| \leqslant 1, \qquad Y: y^{\cdot} = v, \quad |v| \leqslant 1$$
$$x_1(0) = x_1^{\circ}, \quad x_2(0) = x_2^{\circ}; \qquad y(0) = y^{\circ} \tag{1.1}$$

Here x_1, x_2, u, y, v are vectors of arbitrary like dimension . Player X has the following information available to him. At each instant $t \in [0, T]$ he knows the exact value of the natural phase coordinate vectors $x_1(t)$, $x_2(t)$. Player X observes the opponent's phase vector $y(t)$ at N instants a_i, $i = 1, \ldots, N$, $0 = a_1 \leqslant \ldots \leqslant a_N = T$. The observation instants a_i are taken to be fixed. Thus, at the instant $t \in [0, T]$ player X knows the set of quantities $\{t, x_1(t), x_2(t), y(t'(t))\}$, where $t'(t)$ is the last observation instant

$$t'(t) = a_i, \quad t \in [a_i, a_{i+1}), \qquad i = 1, \ldots, N-1 \tag{1.2}$$

We assume that the player X forms a control vector at the instant t, using the available information, i.e. applies a strategy in the form of the function

$$u[t] = u(t, x_1(t), x_2(t), y(t'(t))), \ |u[t]| \leqslant 1, \ t \in [0, T].$$

The aim of player X is to minimize the functional

$$J = |x_1(T) - y(T)| \tag{1.3}$$

Player Y opposes X's intention and realizes his own (admissible) control in the form of an integrable time function $v(t)$, $|v(t)| \leqslant 1$, $t \in [0, T]$. The absolutely continuous functions $x_1(t)$, $x_2(t)$, $y(t)$, $x_1(0) = x_1^\circ$, $x_2(0) = x_2^\circ$, $y(0) = y^\circ$, which satisfy the equations

$$y^\cdot(t) = v(t), \ x_1^\cdot(t) = x_2(t), \ x_2^\cdot(t) = u(t, x_1(t), x_2(t), y(t'(t)))$$

almost everywhere on $[0, T]$, are said to be the solution of system (1.1), corresponding to control v, strategy u and the initial vectors in (1.1). Those strategies which determine a unique solution of system (1.1) for given initial vectors and an admissible control $v(t)$ are taken as the admissible strategies of player X.

Problem 1. Find the optimal minimax strategy u^* of player X, i.e. the strategy satisfying the relation

$$J^* = \min_u \sup_v J[u, v] = \sup_v J[u^*, v] \tag{1.4}$$

Find the minimal value J^* of functional (1.3) guaranteed for player X.

Here $J[u, v]$ is the value of functional (1.3) on the solution of system (1.1), determined by strategy u and control v. The dependence of $J[u, v]$ on the initial vectors is not explicitly indicated. The "min" and "sup" operations are carried out over the sets of admissible strategies and controls. We note that under more general constraints on the control vectors $|u| \leqslant \mu$, $|v| \leqslant \nu$, $\mu, \nu > 0$, the game being considered can be reduced to the form (1.1), (1.3) by the substitution

$$x_1 \to x_1\nu^2/\mu, \quad x_2 \to x_2\nu, \quad u \to u\mu, \quad y \to y\nu^2/\mu,$$
$$v \to v\nu, \quad t \to t\nu/\mu, \quad J \to J\nu^2/\mu.$$

2. The equation of motion (1.1) can be reduced to the form

$$\begin{gathered} X: x^\cdot = (T-t)u, \quad |u| \leqslant 1, \qquad Y: y^\cdot = v, \quad |v| \leqslant 1 \\ x(0) = x^\circ = x_1^\circ + Tx_2^\circ, \qquad y(0) = y^\circ \end{gathered} \tag{2.1}$$

by means of the variable $x(t) = x_1(t) + (T-t)x_2(t)$. To solve Problem 1 using Eqs. (2.1) we assume that player X, by observing the quantities $\{t, x(t), y(t'(t))\}$ at the instant $t \in [0, T]$, realizes a strategy in the form of the function $u[t] = u(t, x(t), y(t'(t)))$. The solution of system (2.1) and the admissible strategies are defined analogously as in Sect. 1. In general, the class of admissible strategies is narrower than the class described in Sect. 1 because the dependency of the strategies on the vectors $x_1(t), x_2(t)$ has been replaced by a dependency on their linear combination $x(t) =$

$x_1(t) + (T - t)x_2(t)$. However, from what follows we shall see that the extremum (1.4) is achieved on a strategy u^* from this narrow class.

Having noted that $x(T) = x_1(T)$, we rewrite functional (1.3) as

$$J = | x(T) - y(T) | \tag{2.2}$$

We introduce the notation $x_k = x(a_k)$, $y_k = y(a_k)$, $k = 1, \ldots, N$. The choice of a certain strategy by player X is equivalent to choosing the collection of functions $u_k(x_k, y_k; t)$, $t \in [a_k, a_{k+1})$, $k = 1, \ldots, N-1$, or, in other words, is equivalent to this that at the instant a_k player X, depending on the position $\{x_k, y_k\}$ realized, specifies beforehand his own control on the interval $[a_k, a_{k+1})$ in the form of an integrable function of time. We denote the collection mentioned by $\{u_k(x_k, y_k; t)\}$ and also call it a strategy. We integrate on the intervals $[a_k, a_{k+1}]$ certain admissible control and admissible strategy implicit in (2.1)

$$\begin{gathered} x_{k+1} = x_k + (a_{k+1} - a_k)[T - a_k - {}^1/_2(a_{k+1} - a_k)]\, u_k, \quad |u_k| \leqslant 1 \\ y_{k+1} = y_k + (a_{k+1} - a_k)\, v_k, \quad |v_k| \leqslant 1, \qquad k = 1, \ldots, N-1 \\ x_1 = x^\circ, \qquad y_1 = y^\circ \end{gathered} \tag{2.3}$$

The vectors v_k and $u_k = u_k(x_k, y_k)$ are determined by the equalities

$$v_k = (a_{k+1} - a_k)^{-1} \int_{a_k}^{a_{k+1}} v(t)\, dt,$$

$$u_k(x_k, y_k) = (a_{k+1} - a_k)^{-1} \alpha_k^{-1} \int_{a_k}^{a_{k+1}} (T - t)\, u_k(x_k, y_k; t)\, dt$$

$$\alpha_k = T - a_k - {}^1/_2(a_{k+1} - a_k), \qquad k = 1, \ldots, N-1$$

On the other hand, for every sequence of $v_k, u_k = u_k(x_k, y_k)$, $|v_k| \leqslant 1$, $|u_k| \leqslant 1$, we can find a player Y's control and a player X's strategy realizing in (2.1) the same values of vectors x_k, y_k, $k = 1, \ldots, N$ as in (2.3). We cite an example of such a control and strategy

$$v(t) = v_k, \quad u_k(x_k, y_k; t) = u_k(x_k, y_k) \quad t \in [a_k, a_{k+1}) \tag{2.4}$$

Instead of the differential game (2.1), (2.2) with inclomplete information we consider the multistage game (2.3) in which player X applies strategies $u_k = u_k(x_k, y_k)$, $k = 1, \ldots, N-1$, and minimizes the functional

$$J = |x_N - y_N| \tag{2.5}$$

If $\{u_k^*(x_k, y_k)\}$ is the optimal minimax strategy in game (2.3), (2.5), then the strategy (2.4) corresponding to it is the optimal minimax strategy in the game (2.1), (2.2), i.e. in correspondence with the remark made, is a solution of Problem 1.

To solve game (2.3), (2.5) we define Bellman's function by the relation

$$\begin{gathered} S_k(x_k, y_k) = \min_{u_k} \max_{v_k} \ldots \min_{u_{N-1}} \max_{v_{N-1}} J, \quad k = 1, \ldots, N-1 \\ S_N(x_N, y_N) = |x_N - y_N| \end{gathered} \tag{2.6}$$

Function (2.6) equals the minimum value of functional (2.5) which can be guaranteed on the trajectories of system (2.3) under the condition that the position $\{x_k, y_k\}$ is realized at the kth stage. From (2.6) follows the recurrence relation

$$S_k(x_k, y_k) = \min_{u_k} \max_{v_k} S_{k+1}, (x_{k+1}, y_{k+1}) \tag{2.7}$$

$$|u_k| \leqslant 1, \ |v_k| \leqslant 1, \qquad k = 1, \ldots, N-1$$

with the boundary condition

$$S_N(x_N, y_N) = |x_N - y_N| \tag{2.8}$$

The vectors x_{k+1}, y_{k+1} in (2.7) should be taken in the form (2.3). By starting from the boundary conditions (2.8) and by computing in successive stages the extremum in (2.7), we can find the unique function satisfying (2.7), (2.8)

$$S_k(x_k, y_k) = \max\left[\Phi_k(a_k, \ldots, a_N), \ |x_k - y_k| + (T - a_k)\left(1 - \frac{T - a_k}{2}\right)\right]$$

$$\Phi_k(a_k, \ldots, a_N) = \max_{k \leqslant i \leqslant N-1} [T - a_i - {}^1/_2(T - a_{i+1})^2], \qquad k = 1, \ldots, N-1$$

The optimal minimax strategy obtained during the computation of the minimum in (2.7) is:

$$u_k^*(x_k, y_k) = (y_k - x_k) / |x_k - y_k|, \ |x_k - y_k| \geqslant (a_{k+1} - a_k)\alpha_k$$

$$u_k^*(x_k, y_k) = (y_k - x_k) / (a_{k+1} - a_k)\alpha_k, \quad |x_k - y_k| < (a_{k+1} - a_k)\alpha_k$$

$$\alpha_k = T - a_k - {}^1/_2(a_{k+1} - a_k), \qquad k = 1, \ldots, N-1 \tag{2.9}$$

Then, the strategy $\{u_k^*(x_k, y_k; t)\}$, obtained substituting (2.9) into (2.4), is a solution, generally nonunique, of Problem 1. Another solution of Problem 1 in the original notation has the form

$$u^* = (y(t') - x(t)) / |x(t) - y(t')|, \qquad x(t) \neq y(t')$$

$$u^* = 0, \quad x(t) = y(t'), \quad x(t) = x_1(t) + (T - t)x_2(t) \tag{2.10}$$

It is not difficult to verify that the two solutions of Problem 1 indicated realize in (2.1) one and the same sequence of values $x_k = x(a_k)$, $k = 1, \ldots, N$. The minimal guaranteed value (1.4) of functional (1.3) (or equivalently, functionals (2.2), (2.5)) is

$$J^* = S_1(x^\circ, y^\circ) = \max[\Phi_1(a_1, \ldots, a_N), \ |x^\circ - y^\circ| + T(1 - T/2)]$$

$$\Phi_1(a_1, \ldots, a_N) = \max_{1 \leqslant i \leqslant N-1} [T - a_i - {}^1/_2(T - a_{i+1})^2] \tag{2.11}$$

3. Above, the observation points a_i were taken as fixed. Let us now pose the problem of the optimal distribution of observation instants, i.e. such at which the minimum value of functional (2.11), guaranteed for player X, is minimal. From (2.11) we see that the optimal distribution a_i^* should be sought from the condition

$$\Phi_1^* = \min_{\{a_i\}} \Phi_1(a_1, \ldots, a_N) \tag{3.1}$$

$$0 = a_1 \leqslant a_2 \leqslant \ldots \leqslant a_N = T$$

From the lemma stated in Sect. 5 it follows that the minimum in (3.1) exists and is achieved on a set of a_i^*, $i = 1, \ldots, N$, such that

$$T - a_i^* - {}^1/_2(T - a_{i+1}^*)^2 = 2h_N(T) = \Phi_1^*, \quad i = 1, \ldots, N-1$$

$$0 = a_1^* < a_2^* < \ldots < a_N^* = T \tag{3.2}$$

Here $h_N(T) > 0$ is an as yet unknown constant depending on the problem parameters N and T. Eliminating the a_i^* from equalities (3.2) we can obtain that $h_N(T)$ equals the only positive root of the equation

$$\beta_N(\xi) \equiv \xi(\xi(\xi\ldots(\xi(\xi+1)^2+1)^2+\ldots+1)^2+1) = T/2, \quad \xi>0, (N-2 \text{ brackets})$$

$$\beta_N(h_N(T)) = T/2 \tag{3.3}$$

The optimal observation instants are computed by the formulas

$$a_i^* = T - 2\beta_{N-i+1}(h_N(T)), \qquad i=1,\ldots,N \tag{3.4}$$

where $\beta_k(\xi)$ are polynomials of degree 2^{k-2} $(k \neq 1)$

$$\beta_k(\xi) \equiv \xi(\xi(\xi\ldots(\xi(\xi+1)^2+1)^2+\ldots+1)^2+1), \quad k=2,3,\ldots (k-2 \text{ brackets})$$

$$\beta_1(\xi) \equiv 0, \quad \beta_{k+1}(\xi) = \beta_k^2(\xi)+\xi, \quad k=1,2\ldots \tag{3.5}$$

The relations

$$\beta_{k+1}(\xi) > \beta_k(\xi), \qquad \xi>0, \ k=1,2,\ldots$$
$$\lim\beta_k(\xi) = \beta(\xi) = {}^1/_2(1-\sqrt{1-4\xi}), \quad k\to\infty, \ 0\leqslant\xi\leqslant{}^1/_4 \tag{3.6}$$
$$\lim\beta_k(\xi) = \infty, \quad k\to\infty, \quad \xi>{}^1/_4$$

are an immediate consequence of (3.5).

Observations at the instants (3.4) guarantee player X the functional value

$$J^* = \max\,[2h_N(T), \ |x^\circ - y^\circ| + T(1-T/2)] \tag{3.7}$$

Let us study the asymptotic behavior of $h_N(T)$ as $N\to\infty$. From the monotony of polynomials $\beta_k(\xi)$ with respect to the index k in (3.6) follows the monotonic decrease

$$h_{N+1}(T) < h_N(T), \quad h_N(T)>0, \quad T>0, \quad N=2,3,\ldots$$

and further the existence of the finite limit $\lim h_N(T) = h(T)$ as $N\to\infty$. The functions $h_N(T)$, $T>0$ and $2\beta_N(\xi)$, $\xi>0$, $N=2,3,\ldots$, are mutually inverse, therefore, from (3.6) we can obtain

$$h(T) = {}^1/_2 T(1-T/2), \qquad 0<T\leqslant 1 \tag{3.8}$$
$$h(T) = {}^1/_4, \qquad T>1$$

Thus, by choosing a sufficiently large number of observation points player X can guarantee a functional value arbitrarily close to

$$J^* = \max\,[2h(T), \ |x^\circ - y^\circ| + T(1-T/2)], \qquad T>0$$
$$J^* = |x^\circ - y^\circ| + T(1-T/2), \qquad 0<T\leqslant 1 \tag{3.9}$$
$$J^* = \begin{cases} {}^1/_2, & |x^\circ-y^\circ| \leqslant {}^1/_2 - T(1-T/2) \\ |x^\circ-y^\circ| + T(1-T/2), & |x^\circ-y^\circ| > {}^1/_2 - T(1-T/2), \quad T>1 \end{cases}$$

The value (3.9) is the minimum guaranteed value of the functional under continuous observation (see [2]). From the initial positions for which we succeed, by choosing N, in obtaining in (3.7) a value of $2h_N(T)$ not exceeding the right-hand expression within the brackets, we can guarantee the exact value (3.9) by an observation only at a finite number of points. This can be achieved from the position $\{x^\circ, y^\circ\}$

$$|x^\circ - y^\circ| > 2h(T) - T(1-T/2), \qquad T>0 \tag{3.10}$$

The minimum number $N(x^\circ, y^\circ)$ of observation points, sufficient to achieve result (3.9) from positions (3.10), equals the minimum integer N satisfying the inequality

$$2h_N(T) \leqslant |x^\circ - y^\circ| + T(1 - T/2)$$

The assertions made follow from (3.7) and from the monotonic tending of $h_N(T)$ to $h(T)$. From the remaining positions

$$|x^\circ - y^\circ| \leqslant 2h(T) - T(1 - T/2) \qquad (3.11)$$

it is impossible to guarantee the value (3.9) by observation at a finite number of points. Here observations are necessary on a countable set of points, whose construction will be presented below.

We define the limit set of observation points $A_T \subset [0, T]$, $A_T = \{a_i^\circ\}$, $a_i^\circ = T - C_i'^\circ$, by the relations

$$c_i^\circ = \lim_{N\to\infty} 2\beta_{N-i+1}(h_N(T)) = 2(\ldots(T/2 - h(T))^{1/2}\ldots - h(T))^{1/2} \quad (i-1 \text{ bracket})$$
$$i = 1, 2, \ldots$$
$$c_{-i}^\circ = \lim_{N\to\infty} 2\beta_{i+1}(h_N(T)) = 2\beta_{i+1}(h(T)), \qquad i = 0, 1, 2, \ldots \qquad (3.12)$$

The first relation in (3.12) is obtained by using properties (3.5) of the polynomials $\beta_k(\xi)$. As we see from (3.12), the point a_i°, $i > 0$, is the limit to which the ith optimal observation point (3.4) tends as $N \to \infty$; the point a_i°, $i < 0$, is the limit to which the ith, counting from the last observation point $a_0^\circ = T$, optimal observation point tends. Passing to the limit in equalities (3.2) as $N \to \infty$, we obtain the recurrence relations for the points a_i°

$$a_1^\circ = 0, \quad a_0^\circ = T; \quad T - a_i^\circ - {}^1/_2(T - a_{i+1}^\circ)^2 = 2h(T), \quad i = 0, \pm 1, \pm 2, \ldots \qquad (3.13)$$

Let us study the condensation points of set A_T. At first let $0 < T \leqslant 1$. From (3.12) and (3.8) it follows that $c_i^\circ = T$, i.e. $a_i^\circ = 0$, $i = 1, 2, \ldots$. Using (3.12), (3.6), (3.8) we find $\lim c_{-i}^\circ = 2\beta(h(T)) = T$, $0 < T \leqslant 1$, $i \to +\infty$, i.e. the set A_T has a single condensation point $a_1^\circ = 0$. The penultimate observation point is the point a_{-1}°

$$a_{-1}^\circ = T - c_{-1}^\circ = T - 2\beta_2(h(T)) = T - T(1 - T/2) = T^2/2 \qquad (3.14)$$

We consider the case $T > 1$. Passing to the limit in (3.13) as $i \to \pm\infty$, we obtain $\lim a_i^\circ = \lim a_{-i}^\circ = \alpha$, $i \to +\infty$, and the equation for α

$$T - \alpha - {}^1/_2(T - \alpha)^2 = {}^1/_2$$

from which the unique value $\alpha = T - 1$ is determined. Consequently, the single condensation point of set A_T, $T > 1$ is the point $T - 1$. The second and the penultimate observation instants a_2° and a_{-1}° are determined from (3.12)

$$a_2^\circ = T - c_2^\circ = T - \sqrt{2T - 1}, \qquad a_{-1}^\circ = T - c_{-1}^\circ = T - {}^1/_2 \qquad (3.15)$$

It can be shown that a strategy of form (2.10), where

$$t'(t) = a_i^\circ, \quad t \in [a_i^\circ, a_{i+1}^\circ), \qquad i = \pm 1, \pm 2, \ldots$$

defines the unique solution of system (1.1) and guarantees the value (3.9) of the functional for any initial position and any admissible control $v(t)$.

The set $A \subset [0, T]$ of instants is called a sufficient set of observation instants for the position $\{x^\circ, y^\circ\}$ if the observations of player X on set A guarantee the results of the game with continuous observation, i.e. the value (3.9) of the functional. It was shown above that for positions (3.10) there exists a minimal sufficient set $A = \{a_1^*, a_2^*, \ldots, a_N^*\}$, where the number N of observation points was determined as a function of the position. For example, the set A_T of (3.12) is a sufficient set for positions (3.11). Here the set A_T is not a unique sufficient set. Among other sufficient sets with a countable number of observation points the set A_T is distinguished by the fact that every two observation instants following one after the other are separeted by the maximum distance. We remark also that in another arbitrary sufficient set of observation instants the points $t = T - 1$, $T > 1$, $t = 0$, $0 < T \leqslant 1$ are condensation points.

From the structure of set A_T it follows that observations on the initial stage of the motion are most important for $0 < T \leqslant 1$, while the observations on the last segment of the motion of duration $T(1 - T/2)$ can be completely omitted (see (3.14)). A finite number of observation points remain outside any arbitrary small interval $[0, \varepsilon]$, $\varepsilon > 0$. For $T > 1$ the point $T - 1$ serves as a condensation point of set A_T, therefore, the observations carried out at one unit of time to the end of the motion, are most important. A finite number of observation points are distributed outside any arbitrarily small interval $[T - 1 - \varepsilon, T - 1 + \varepsilon]$, $\varepsilon > 0$. Further, observations can be omitted on the initial segment of motion of duration $T - \sqrt{2T - 1}$ and on the last segment of duration $1/2$ (see (3.15)).

In order to clarify the reason of distinction of the instant $t = T - 1$ from the remaining instants, we consider game (1.1), (1.3) under the more general constraints: $|u| \leqslant \mu$, $|v| \leqslant \nu$. From the remarks at the end of Sect. 1 it follows that the point $t = T - \nu / \mu$ is then the point of condensation of set A_T, $T > \nu / \mu$. The ratio ν / μ determines the length of the interval on the last segment of motion during which the active encounter of player X with player Y is hampered because of the inertialess behavior of the latter. Therefore, the observations carried out before the weakly controllable segment of motion are important for player X.

4. Let us show that by observation at no more than three instants the player Y in game (1.1), (1.3) can guarantee the maximum value of functional (1.3), equal to (3.9). This, in particular, justifies the contraction of the class of admissible strategies of player X, carried out in Sect. 2. In (1.1) let player Y observe the position $\{t, x_1(t), x_2(t), y(t)\}$ at the three instants $0 = a_1 \leqslant a_2 \leqslant a_3 = T$ and let him strive to maximize the functional (1.3). The admissible control of player X, the admissible strategy of player Y and the solution of system (1.1) we define analogously as in Sect. 1.

P r o b l e m 4. Find the optimal maximin strategy v_* of player Y, i.e. the strategy satisfying the relation

$$J_* = \max_v \inf_u J[u, v] = \inf_u J[u, v_*] \tag{4.1}$$

Find the maximum value J_* of functional (1.3) guaranteed for player Y.

As above, Problem 4 can be reduced to an equivalent multistage game of form (2.3). The Bellman function for this multistage game satisfies the relation of type (2.7) in which the order of the minimum and maximum operations is changed. Having solved the similar relation, we can obtain an expression for the guaranteed maximum (4.1)

$$J_* = \max\left[0,\ (T - a_2)\left(1 - \frac{T - a_2}{2}\right),\quad |x^\circ - y^\circ| + T\left(1 - \frac{T}{2}\right)\right] \tag{4.2}$$

and the strategy guaranteeing this value (in terms of the original problem)

$$v_* = (x(t') - y(t)) / |x(t) - y(t)|, \quad x(t') \neq y(t)$$
$$v_* = e, |e| = 1, x(t') = y(t), x(t) = x_1(t) + (T - t)x_2(t)$$

The instant t' (t) has been defined in (1.2) with $N = 3$. We see that having set $a_2 = a_1 = 0$ for $0 < T \leqslant 1$ and $a_2 = T - 1$ for $T > 1$ in (4.2), we obtain the very same value (3.9) for J_*. Thus, in order that a saddle situation would take place in game (1.1), (1.3), it is necessary and sufficient for player Y to make observations at two $(0 < T \leqslant 1)$ or at three $(T > 1)$ points; it is sufficient for player X to make observations at a finite or countable set of points, constructed in Sect. 3. In those cases when it suffices for player X to observe at a finite number of points, the question of the minimum information needed for realizing the saddle situation, can be discussed.

5. Lemma. Let a function $g(x, y)$ be continuous in the closed square $a \leqslant x, y \leqslant b$ and differentiable in the open square $a < x, y < b$, and let

$$\partial g / \partial x < 0, \quad \partial g / \partial y > 0, \quad a < x, y < b \tag{5.1}$$

Let $z = (z_1, \ldots, z_n)$ be an n-dimensional vector, $n > 2$ and let $z \in G$ signify that: $z_1 = a$, $z_n = b$, $a \leqslant z_i \leqslant b$, $i = 2, \ldots, n - 1$. Then, the minimum

$$h = \min_{z \in G} \Phi(z), \quad \Phi(z) = \max_{1 \leqslant i \leqslant n-1} g(z_i, z_{i+1})$$

is achieved at a single point $z^* \in G$ such that

$$h = g(z_i^*, z_{i+1}^*), \quad i = 1, \ldots, n-1, \quad a = z_1^* < z_2^* < \ldots < z_n^* = b \tag{5.2}$$

Let us sketch the proof. The desired minimum is achieved since the set G is closed and the function $\Phi(z)$ is continuous on it. If we assume that equality (5.2) is not fulfilled, then, by using (5.1), we can construct a variation δz of vector z^*, $z^* + \delta z \in G$, such that $\Phi(z^* + \delta z) < \Phi(z^*)$. Finally, the uniqueness and the strict monotonic growth of the coordinate vector z^* can be proved by taking equalities (5.2) as the equations for determining the successive coordinate z_{i+1}^* when the preceding one z_i^* is known.

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